WS 2016/17

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Symplectic Geometry

Homework 5

Exercise 1. (8 points)
Recall that the complex projective space $\mathbb{C}P^n$ is a real $2n$-dimensional manifold

$$\mathbb{C}P^n := \big\{(z_0,\dots,z_n)\in\mathbb{C}^{n+1}\setminus\{0\}\big\}/\sim,$$

where $(z_0,\dots,z_n)\sim(\lambda z_0,\dots,\lambda z_n)$ for any $\lambda\in\mathbb{C}\setminus\{0\}$. Show that

$$H^k_{dRham}(\mathbb{C}P^n)=\begin{cases}\mathbb{R} & \text{if } k=0,2,\dots,2n\\ 0 & \text{otherwise}\end{cases}.$$

Note that the theorems from the lecture already imply that $H^0_{dRham}(\mathbb{C}P^n)=H^{2n}_{dRham}(\mathbb{C}P^n)=\mathbb{R}$.
Hint: You may want to use the Mayer-Vietoris Sequence with $U=\{[(z_0,\dots,z_n)]\in\mathbb{C}P^n,\ z_n\neq 0\}$ and $V=\mathbb{C}P^n\setminus\{[(0,\dots,0,1)]\}$.

Exercise 2. (3 points)
Find the constant $a\in\mathbb{C}$ such that for the identification $\Psi\colon\mathbb{C}^n\to\mathbb{R}^{2n}$ given by $(z_1,\dots,z_n)=(x_1+iy_1,\dots,x_n+iy_n)\mapsto(x_1,\dots,x_n,y_1,\dots,y_n)$ one has that $\Psi^*(\sum_{j=1}^n dx_j\wedge dy_j)=a\sum_{j=1}^n dz_j\wedge d\bar{z}_j$.

Exercise 3. (5 points)
Observe that for $S^2\subset\mathbb{R}^3$ one can identify T_pS^2 with $\{v\in\mathbb{R}^3;\ \langle p,v\rangle=0\}$. Show that the 2-form ω on S^2 given by

$$\omega_p(v_1,v_2):=\langle p, v_1\times v_2\rangle,$$

(where $v_1\times v_2$ is the cross product in $\mathbb{R}^3$), is symplectic.

Exercise 4. (8 points)
Consider cylindrical polar coordinates (θ,x_3) on the sphere minus its poles $S^2\setminus\{(0,0,\pm1)\}$, where $0\le\theta<2\pi$ and $-1<x_3<1$. Show that the area form induced by the Euclidean metric is precisely the from $\omega=d\theta\wedge dx_3$. In other words, the horizontal projection from the cylinder to the sphere preserves the surface area.

Bonus Exercise. (8 points)
Let $(q_1, \ldots, q_n) \colon U \to \mathbb{R}^n$ be a coordinate chart for an n-dimensional manifold X. Recall that then any 1-form $\xi \in \Omega^1(X)$ is locally given by

$$\xi = \sum_{j=1}^{n} p_j dq_j$$

and $(q_1, \ldots, q_n, p_1, \ldots, p_n) \colon T^*U \to \mathbb{R}^{2n}$ is a coordinate chart for T^*X. The tautological 1-form $\alpha \in \Omega^1(T^*X)$ on T^*X is locally defined as

$$\alpha_{|(q,p)} = \sum_{j=1}^{n} p_j dq_j.$$

Show that α is intrinsically defined, i.e. that this definition does not depend on the choice of coordinate chart.

Exercise 5. (8 points)
Show that the above α, at a point $v \in T_q^*X \subset T^*X$ can also be defined by

$$\alpha_{|v} = (d\pi_v)^* v,$$

where $d\pi_v \colon T_v(T^*X) \to T_qX$ is the derivative (push-forward) at v of the projection map $\pi \colon T^*X \to X$ and $(d\pi_v)^*$ is its transpose. (In local coordinates, $d\pi(\frac{\partial}{\partial q_j}) = \frac{\partial}{\partial q_j}$ and $d\pi(\frac{\partial}{\partial p_j}) = 0$.)

Exercise 6. (8 points)
Show that the tautological 1-form α is uniquely characterized by the property that, for every 1-form $\mu \colon X \to T^*X$, it holds that the pull back $\mu^*\alpha$ of α via the map μ is equal to μ itself.

Bonus Exercise. (8 points)
Show that there is an isomorphism

$$T_{(q,0)}T^*X \cong T_qX \oplus T_q^*X$$

and

$$-d\alpha_{(q,0)}(v, w) = w_1^*(v_0) - v_1^*(w_0)$$

for $v = (v_0, v_1^*) \in T_qX \oplus T_q^*X$ and $w = (w_0, w_1^*) \in T_qX \oplus T_q^*X$.

Hand in: Thursday November 24th
in the exercise session
in Übungsraum 1, MI